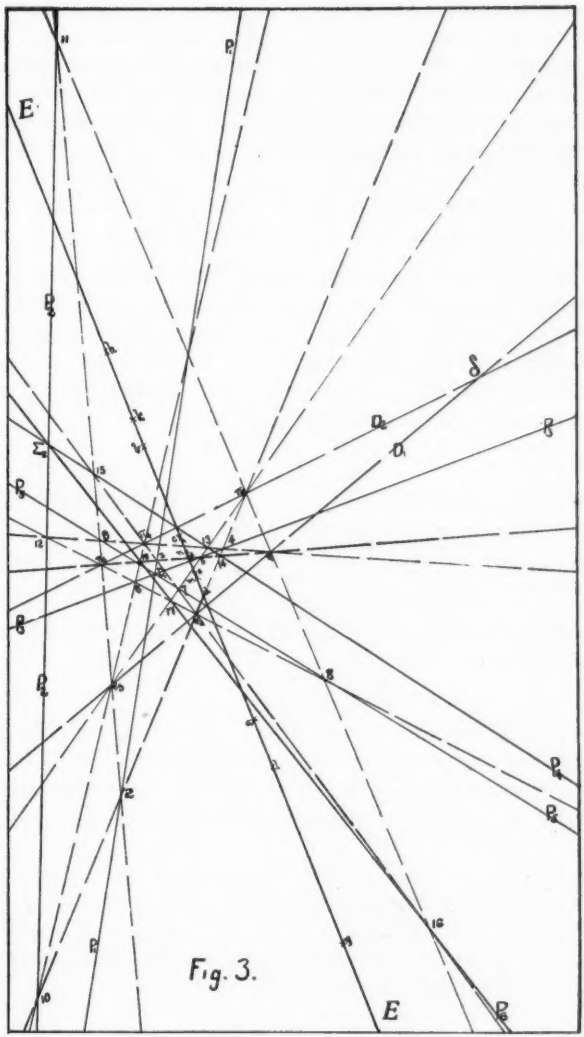


Fig. 1.

The triangles of each set are independently by twos in triple perspective, having as centers of perspective the points Σ_1 and Σ_2 each three lines and three points π_i on the line D_2 or D_1, respectively; and

The triangles of each set are independently by twos in triple perspective, having as axes of perspective the lines D_1 and D_2 each three times and three lines P_i on the point Σ_2 or Σ_1, respectively; and having

having as axes of perspective the line D_1 or D_2 three times each for the sets, respectively, and twelve other lines all of which pass through a point ε, which is the pole of the line S as to any of the six triangles. The 18 lines above lie by three on six points π_i, which are three and three on the lines D above.

as axes of perspective the point Σ_1 or Σ_2 three times each for the sets, respectively, and twelve other points all of which lie on a line E, which is the polar of the point δ as to any of the six triangles. The 18 points above lie by three on six lines P_i, which are three and three on the points Σ above.

The frame-work of this theorem as to the triangles and their being in triple perspective is not new but seems a necessary preface to the new parts as to the covariant point ε, and line E, and the complete duality that makes the figures really one whole since starting with the three points π_i on each line D_1 and D_2 of the left-hand theorem and following out the right-hand theorem we come back to the original points Σ_1 and Σ_2.

II.

The lines D_1, D_2, and S are the false sides of the complete quadrilateral of the Hessian pairs of the line-triads P_i on Σ_1 and Σ_2.

The points Σ_1, Σ_2, and δ are the false vertices of the complete quadrangle of the Hessian pairs of the point-triads π_i on D_1 and D_2.

From these theorems as also from the demonstration of them, follows the general theorem:

III.

Three lines on each of two points give rise to three points on each of two lines, and the latter by reciprocating the process give rise to three lines on each of the original two points. The derived three lines have the same Hessian pair, or are inclined to each other at the same angles as the original three.

4. We now give a proof of the theorems on the left. Take the two Steiner points, Σ_1, Σ_2, to have co-ordinates σ_1, σ_2, σ_3 and s_1, s_2, s_3, respectively; the triangle $a_1a_2a_3$, formed by the intersections of the Pappus lines as will be shown, as reference triangle; and the line S as auxiliary line.

The line S determined by Σ_1 and Σ_2 is given by

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = 0, \text{ and will be written}$$

$$S_1x_1+S_2x_2+S_3x_3=0.$$

Since this line is taken as auxiliary line, we have the following relations:

$$S_1=S_2=S_3=1, \quad \sigma_1+\sigma_2+\sigma_3=0, \quad s_1+s_2+s_3=0, \quad s_i^2\sigma_j\sigma_k-s_j s_k\sigma_i^2=\sigma_k s_k-\sigma_j s_j, \tag{3}$$

where i, j, k are each 1, 2, 3, successively.

The equations of the Pappus lines are the corresponding minors as represented thus:

$$\begin{array}{ccc|ccc} P_1 & P_3 & P_5 & P_2 & P_4 & P_6 \\ \sigma_1 & \sigma_2 & \sigma_3 & s_1 & s_2 & s_3 \\ x_1 & x_2 & x_3 & x_1 & x_2 & x_3 \end{array} \tag{4}.$$

The nine intersections of the six lines P other than Σ_1 and Σ_2 are named thus:

$$\begin{array}{rccc|ccc} \text{Line } P_1 & P_5 & P_3 & & & \\ \text{meets line } P_2 & P_4 & P_6, \text{ respectively, in point} & a_2 & a_3 & a_1 \\ P_4 & P_6 & P_2, & \beta_1 & \beta_2 & \beta_3 \\ P_6 & P_2 & P_4, & \gamma_3 & \gamma_1 & \gamma_2 \end{array} \tag{5}.$$

From these co-ordinates by carrying out scheme (1) we find the co-ordinates of the points π_1, π_3, π_5, and of the line D_1 on which they lie. Likewise, following (2) and remembering the relations (3), we find the co-ordinates of the points π_2, π_4, π_6 and of their line D_2. From the co-ordinates of the lines D_1 and D_2 we write the co-ordinates of their intersection.

The 9 lines which by threes pass through the points π_i on D_1 are the sides of three point-triads

$$\left\{\begin{array}{llll} (1) & a_1 & a_2 & a_3, \\ (2) & \beta_1 & \beta_2 & \beta_3, \\ (3) & \gamma_1 & \gamma_2 & \gamma_3; \end{array}\right\} \text{ and } \left\{\begin{array}{llll} (1') & a_1 & \beta_1 & \gamma_1, \\ (2') & a_2 & \beta_2 & \gamma_2, \\ (3') & a_3 & \beta_3 & \gamma_3, \end{array}\right\}$$

are the second set of three triangles whose sides pass by threes through the points π_{2i} on D_2.

The triangles of each set are found to be in triple perspective as indicated thus:

Triangles.	*Centers.*	*Axes of Perspective.*
$\begin{matrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \end{matrix}$	$\pi_2, \Sigma_2, \Sigma_1.$	$D_1, (\sigma_2, \sigma_3, \sigma_1), (s_3, s_1, s_2).$
$\begin{matrix} a_1 & a_2 & a_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{matrix}$	$\pi_4, \Sigma_1, \Sigma_2.$	$D_1, (s_2, s_3, s_1), (\sigma_3, \sigma_1, \sigma_2).$
$\begin{matrix} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{matrix}$	$\pi_6, \Sigma_2, \Sigma_1.$	$D_1, (\sigma_1, \sigma_2, \sigma_3), (s_1, s_2, s_3).$

$$\left.\begin{matrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \end{matrix}\right\} \qquad \pi_1,\ \Sigma_2,\ \Sigma_1. \qquad D_2,\ (\sigma_2,\ \sigma_1,\ \sigma_3),\ (s_2,\ s_1,\ s_3).$$

$$\left.\begin{matrix} a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{matrix}\right\} \qquad \pi_3,\ \Sigma_2,\ \Sigma_1. \qquad D_2,\ (s_3,\ s_2,\ s_1),\ (\sigma_3,\ \sigma_2,\ \sigma_1).$$

$$\left.\begin{matrix} a_3 & \beta_3 & \gamma_3 \\ a_1 & \beta_1 & \gamma_1 \end{matrix}\right\} \qquad \pi_5,\ \Sigma_1,\ \Sigma_2. \qquad D_2,\ (\sigma_1,\ \sigma_3,\ \sigma_2),\ (s_1,\ s_3,\ s_2).$$

The twelve axes other than the D's, evidently pass through a point with co-ordinates (1, 1, 1), which is thus the auxiliary point ϵ, the pole S as to the reference triangle. Since,

S has the same pole, ϵ, as to all point-cubics consisting of three joins of the six lines P_i, two and two,

ϵ is the pole of S as to every and any of the six triangles on all six lines P_i.

δ has the same polar, E, as to all line-cubics consisting of three joins of the six points π_i, two and two,*

E is the polar of δ as to every and any of the six triangles on all six points π_i.

THE LINES D_1 AND D_2 AS DIAGONALS OR FALSE SIDES OF THE COMPLETE QUADRILATERAL OF THE HESSIAN PAIRS.

5. The following linear relation exists between the three lines P_i on Σ_1:

$$\sigma_1(\sigma_2x_3-\sigma_3x_2)+\sigma_2(\sigma_3x_1-\sigma_1x_3)+\sigma_3(\sigma_1x_2-\sigma_2x_1)=0.$$

The Hessian covariant of the binary cubic is given by the sum of the squares of these three terms separately, and the imaginary Hessian lines are these terms with the respective coefficients 1, ω, ω^2, and 1, ω^2, ω, in the second case. These two lines and the analogous two on Σ_2 reduce to

$$\sigma_2\sigma_3x_1+\omega\sigma_3\sigma_1x_2+\omega^2\sigma_1\sigma_2x_3=0, \qquad \{1\}$$

$$\sigma_2\sigma_3x_1+\omega^2\sigma_3\sigma_1x_2+\omega\sigma_1\sigma_2x_3=0, \qquad \{2\}$$

The same as {1} with s for σ, {3}

The same as {2} with s for σ, {4}.

These intersect as follows:

{1} and {3} in imaginary point I: $(\sigma_1s_1,\ \omega^2\sigma_2s_2,\ \omega\sigma_3s_3)$.

{2} and {4} in imaginary point J: (same with ω and ω^2 interchanged).

{1} and {4} in imaginary point H: $[(\sigma_1s_1(\sigma_2s_3-\omega\sigma_3s_2),\ \sigma_2s_2(\sigma_3s_1-\omega\sigma_1s_3),\ \sigma_3s_3(\sigma_1s_2-\omega\sigma_2s_1)]$.

{2} and {3} in imaginary point K: [same with ω^2 for ω].

*Salmon: *Higher Plane Curves*, §166, pp. 143, 144.

Whence we find that the line of $\overline{IJ}$ is D_1, and that of $\overline{HK}$ is D_2; therefore, D_1 and D_2 are the diagonals of the imaginary quadrilateral of the Hessian pairs of the line-triads on Σ_1 and Σ_2.

6. As may be seen from Fig. 1, for the reverse process we have the triply perspective triangles, thus:

Triangles.	*Centers.*	*Axes of Perspective.*
1 6 8 / 2 4 9	Σ_1, a, b.	P_2, D_2, D_1.
1 6 8 / 7 3 5	c, Σ_1, d.	D_2, P_6, D_1.
2 4 9 / 5 7 3	e, f, Σ_1.	D_2, D_1, P_4.
10 14 18 / 16 11 15	g, Σ_2, h.	D_2, P_1, D_1.
11 15 16 / 17 19 13	k, Σ_2, l.	D_2, P_5, D_1.
10 14 18 / 12 13 17	Σ_2, m, n.	P_3, D_2, D_1.

7. The special forms or arrangements for the three lines on each of two points, to which attention is called, are:

(1) Two sets of equispaced triads, *i. e.*, lines at 120°.

(a) The points being the two equiangular points of the triangles of one set of three.

(b) The points on the circumcircle of equiangular triads.

(2) The points taken at infinity,

(a) Arbitrarily, giving two sets of three parallel lines each, at an angle D_1 with each other.

(b) At I and J, the circular imaginary points, giving two sets of perpendicular lines.

The several forms are handled best by using different co-ordinate systems most convenient for the particular case.

A special form of three points on each of two lines is had when three of the nine lines cross-joining the points by pairs meet in a point either within or without the acute angle at δ. Evidently one triangle of one or of the other set of three becomes a point, which point is then also Σ_1 or Σ_2 and the first six or the second six of the twelve points on E, which thus passes through the same point. All this follows readily in the analysis by observing the relation existing between the co-ordinates when three of the vertices of one triangle are equated.

8. Without further proof because they follow from data already given and may be tested in Fig. 1, we present two more theorems and two others relative to the special forms above.

IV.

A triangle of one set of three is in two-fold perspective with any one of the opposite set of three triangles, but for all such perspectivities—

there are only nine axes each taken twice and situated on the covariant point ϵ. The centers in each case are Σ_2 and Σ_1.

there are only nine centers each taken twice and situated on the covariant line E. The axes in each case are D_2 and D_1.

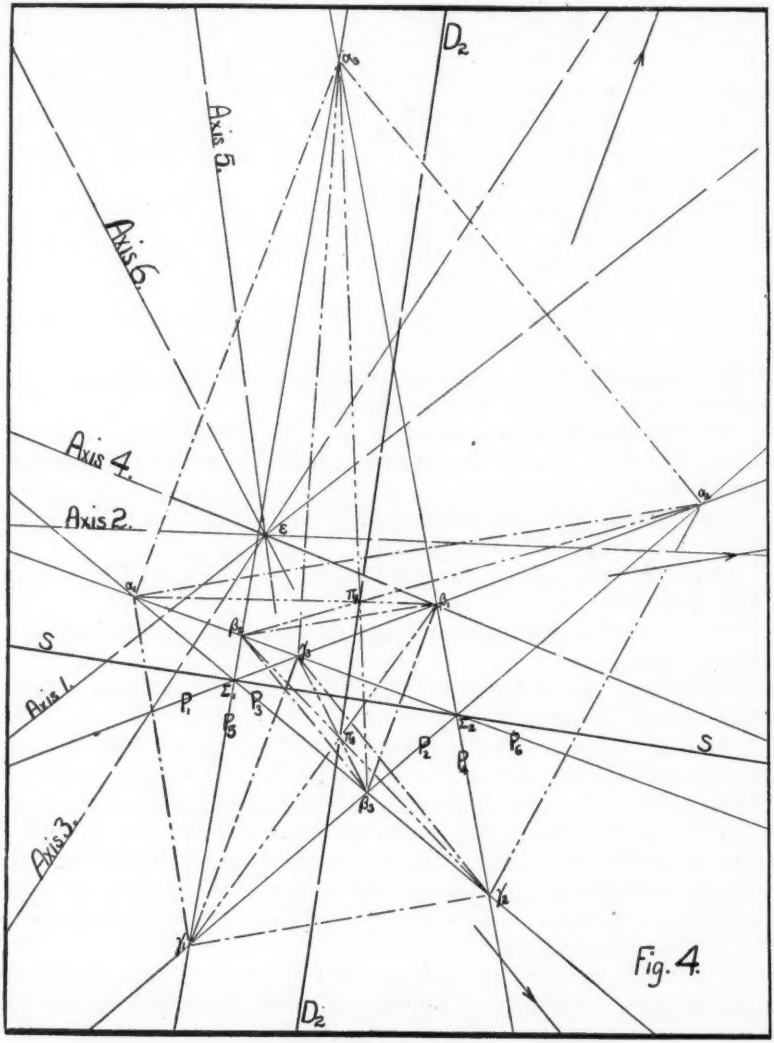


Fig. 2.

V.

The point- and line-triads are between themselves in *single* perspective. The center of perspective in each case is Σ_1 if the two triads are both of the first or both of the second set of three as herein classified, and Σ_2 is center if they are of oppositely named sets.

VI.

For a triad of equispaced lines on each of two points, one set of the three point-triads consists of equiangular triangles with sides respectively parallel (Fig. 2). Thus one of the lines D is at infinity and the other is perpendicular bisector of the line S between Σ_1 and Σ_2. Furthur, the circumcircles of the three equilateral triads pass through Σ_1 and Σ_2, and those of the other set of three triangles intersect in ϵ, the pole of S as to any of the six triangles.

VII.

If to the conditions of the previous theorem we add that one of the set of scalene triangles is also equiangular, then the vertices of the other two triangles of its set are inverse points as to its circumcircle, and all the circumcenters of the set of three equilateral triangles are on the finite Hessian diagonal D.

The proof of this last theorem is especially neat by use of circular co-ordinates.

USING CONJUGATE CO-ORDINATES.

9. As a convenient projection of the hexagon, we take the points π_2, π_4, π_6 on a line, considered the axis of reals, and the points π_1, π_3, π_5 on the line at infinity, so that the lines from the three points on the axis to these three are equispaced lines, parallel, respectively, to

$$\begin{array}{lccc} & x=ty & x=\omega ty & x=\omega^2 ty \\ \text{going, respectively, to} & \pi_1 & \pi_3 & \pi_5 \end{array}$$

$$\begin{array}{lll} \text{Lines on } \pi_2 \equiv a: & x=ty-a(t-1). & x=\omega ty-a(\omega t-1). \quad x=\omega^2 ty-a(\omega^2 t-1). \\ \text{Lines on } \pi_4 \equiv b: & x=ty-b(t-1). & x=\omega ty-b(\omega t-1). \quad x=\omega^2 ty-b(\omega^2 t-1). \\ \text{Lines on } \pi_6 \equiv c: & x=ty-c(t-1). & x=\omega ty-c(\omega t-1). \quad x=\omega^2 ty-c(\omega^2 t-1). \end{array} \tag{6}$$

The points (p), for $p=1, 2, \ldots, 9$, are in general according to the scheme above,

$$\pi_{i,a} \text{ with } \pi_{j,b} \text{ is } x=\frac{a-b\omega-(a-b)\omega^{i+j}t}{1-\omega};$$

and the points (q), where $q=10, 11, \ldots, 18$, are,

$$\pi_{i,a} \text{ with } \pi_{j,b} \text{ is } x=\frac{a-b\omega^2-(a-b)\omega^{i+j-1}t}{1-\omega^2},$$

or are got from points (p) by interchanging b and c in the equation where $p=q-9$. The same interchange holds throughout the paragraph.

In the above a and b each permute for a, b, c, but a is never b; and ij are each 1, 3, 5, but $i \neq j$ in any one equation. As indicated above a, b, c are the circular co-ordinates of π_2, π_4, π_6 along D_2, the axis of reals.

These 18 points in 6 sets of 3 each, as indicated in schemes (1) and (2), lie on six lines P_i. From the equations of points (p) we get the co-ordinates of lines P.

The lines P_i, for $i=1, 3, 5$, we have the co-ordinates

$$P_i : a+b\omega+c\omega^2, \quad -(a+b\omega^2+c\omega)\omega^{-i}t,$$
$$\omega^2(ab+bc\omega+ca\omega^2)-(ab+bc\omega^2+ca\omega)\omega^{-i+1}t.$$

The co-ordinates of lines P_j, for $j=2, 4, 6$, are

$$P_j : a+b\omega^2+c\omega, \quad -(a+b\omega+c\omega^2)\omega^{-j+1}t,$$
$$\omega(ab+bc\omega^2+ca\omega)-(ab+bc\omega+ca\omega^2)\omega^{-j}t.$$

These lines are evidently equispaced on Σ.

The three lines P_i lie on the point Σ_1, which is

$$x=-\omega^2\frac{ab+bc\omega+ca\omega^2}{a+b\omega+c\omega^2};$$

and the lines P_j lie on the point Σ_2,

$$x=-\omega\frac{ab+bc\omega^2+ca\omega}{a+b\omega^2+c\omega}.$$

These two points are evidently conjugate and therefore symmetrical as to the axis of reals. Further, since the axis of reals, D_2, bisects the line between Σ_1 and Σ_2, the two points Σ are harmonic as to the two lines D.

Since the points Σ are independent of t, these points remain the same for any three equispaced lines on a, b, and c, respectively, mutually parallel; or, keeping one triad of points fixed the triad on the other line may move all along that line provided only that the angles between the lines to them remain constant.

Reversing the Process.

10. The lines P_{ij} intersect in nine points, called before α_{1-3}, β_{1-3}, γ_{1-3}, as follows:

$$P_1P_2 : x=\frac{a^2-bc+(a-b)(a-c)\omega^2 t}{2a-b-c}.$$

$$P_1P_4 : x=\frac{c^2-ab+(c-a)(c-b)\omega t}{2c-a-b}.$$

$$P_{16} : x=\frac{b^2-ca+(b-c)(b-a)t}{2b-c-a}.$$

The remaining may be written easily by comparing these with the following:

	P_1	P_2	P_3
P_2	a^2, ω^2	b^2, ω	$c^2, 1$
P_4	c^2, ω	$a^2, 1$	b^2, ω^2
P_6	$b^2, 1$	c^2, ω^2	a^2, ω

The lines joining these intersections in pairs as indicated by schemes (1) and (2) meet by threes in points along D_2 and D_1.

The lines (q) to the *new* points along D_1 have slopes, respectively, t^2, ωt^2, $\omega^2 t^2$; so they turn twice the angle from the axis as the original lines and are like them equispaced.

The lines (p) intersect by threes on three new points, π'_1, π'_3, π'_5, along D_2, corresponding, respectively, with a, b, c if we consider the external segment of D_2. They are

$$x=\frac{\begin{vmatrix} a-b & b^2-ca \\ c-a & c^2-ab \end{vmatrix}}{\begin{vmatrix} a-b & 2b-c-a \\ c-a & 2c-a-b \end{vmatrix}}; \text{ etc.,}$$

taking a, b, c in cyclic order.

11. Thus far the origin on the axis has been arbitrary. Now, considering it the centroid of the three given points, we have $a+b+c=0$, whence also

$$a^2-bc=b^2-ca=c^2-ab=\lambda, \text{ say; } bc+ca+ab=-\lambda; \ 2a-b-c=3a, \text{ etc.}$$

The three new points then become, respectively,

$$x=\frac{a\lambda}{-2bc+ca+ab}, \quad x=\frac{b\lambda}{bc-2ca+ab}, \quad x=\frac{c\lambda}{bc+ca-2ab}.$$

The counter-triad of the three points a, b, c is

$$\frac{-2bc+ca+ab}{3a} \text{ from } (xa/bc)=-1; \text{ call it } a'.$$

$$\frac{bc-2ca+ab}{3b} \text{ from } (xb/ca)=-1; \text{ call it } b'.$$

$$\frac{bc+ca-2ab}{3c} \text{ from } (xc/ab)=-1, \text{ call it } c'.$$

The cubic along the line a, b, c is $x^3-\lambda x-abc=0$, Differentiating this as to x, we have $3x^2-\lambda=0$, the roots of which are the polar pair of infinity, the intersection of the two lines D_1 and D_2. Calling the roots f and f', we have

$$f+f'=0, \text{ and } ff'=-f^2=-f'^2=-\tfrac{1}{3}\lambda. \quad \therefore f^2=f'^2=\tfrac{1}{3}\lambda.$$

By observation we see then that

$$\pi'_1\ a'=\pi'_3\ b'=\pi'_5\ c'=\tfrac{1}{3}\lambda;$$

so the new triad and the counter-triad of the original triad are in an involution whose double points are the polar pair of the intersection of the lines D_1 and D_2 as to the original triad.

From the results in § 9—11, we see that *in this form* by revolving the three lines on Σ, keeping the triad equispaced, we cut out, at each instant, along D_2 a triad of points having the same Hessian pair and being in the same involution whose double-points are the polar pair of the intersection of the Hessian diagonals as to the original triad.

Thus, we generate a pencil of point-triads along each of two lines, and dualistically of line-triads on each of two points.

THE TRISECTION PROBLEM.

By J. S. BROWN, Southwest Texas State Normal School, San Marcos, Texas.

The solution of this problem by means of the quadratrix, conchoid, and the cardioid are well known, and statements of the fact that the problem has been solved by means of the hyperbolic curve are equally well known, but the writer has never seen a solution by the last named method.

Ball, in his History of Mathematics, says that Viviani solved the problem by means of the *equilateral* hyperbola, and that Vieta determined that its solution depends upon the solution of a cubic equation.

I am not aware that the solution by means of the ceroid [so called from its resemblance to a pair of horns] has ever before been given.

I. Solution by Means of the Hyperbolic Curve.

If a series of circles be drawn through two points A and B, and if BP be one third of the arc BPA and H and H' points in the perpendicular bisector of AB, the locus of the point P, as the circle varies in size, is an hyperbola, since $PB=2PH$ constantly.

As the curve is central, its general equation is

$$a^2y^2-b^2x^2=-a^2b^2, \qquad (1).$$

Dividing the members of this equation by b^2 puts it in the form

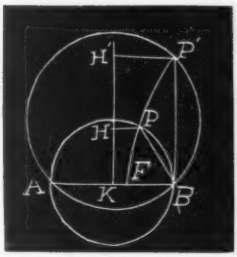
$$\left(\frac{a^2}{b^2}\right)y^2-x^2=-a^2, \qquad (2).$$

The factor $\frac{a^2}{b^2}=\cot^2\theta$, θ being the angle which the asymptote makes with the transverse axis. Therefore (2) becomes

$$y^2\cot^2\theta-x^2=-a^2, \qquad (3).$$

Since e (eccentricity)$=2$, $\cot^2\theta=\frac{1}{3}$ and θ is therefore 60°. Substituting this value of θ in (3) gives

$$\tfrac{1}{3}y^2-x^2=-a^2, \qquad (4).$$



Assuming AB as unity, the circle ABP' as one in which AB is the side of an inscribed square and K the mid point of AB as origin, $P'B=AB=1$, which is the ordinate of the point P', and the abscissa of this point is $a+\frac{1}{2}$.

Substituting these values of y and x in (4), gives

$$\tfrac{1}{3}-(a+\tfrac{1}{2})^2=-a^2, \qquad (5),$$

from which $a=\frac{1}{3}$. This value of a substituted in (4) gives $y^2-3x^2=-\frac{1}{3}$, the equation of the above described curve.

To trisect an angle by means of this curve, construct an isosceles triangle upon AB as a base, with the given angle as vertical angle.

The hyperbola will trisect the arc of the circle whose center is the vertex of this angle and whose radius is the leg of the triangle constructed.

II. Solution by Means of the Ceroid.

If a line be drawn through the center O of a circle to meet the circumference and also to meet a straight line LL', let us find the locus of the point P on HO, is HP is constantly equal to KP. Let O be the origin, d the distance from O to LL', $PH=PK=n$, and let r=the radius of the circle. We have

$$\frac{y}{d}=\frac{r+n}{r+2n}, \qquad (1),$$

from which is obtained the equation,

$$\frac{y^2}{d^2}(r+2n)^2=x^2+y^2, \tag{2}$$

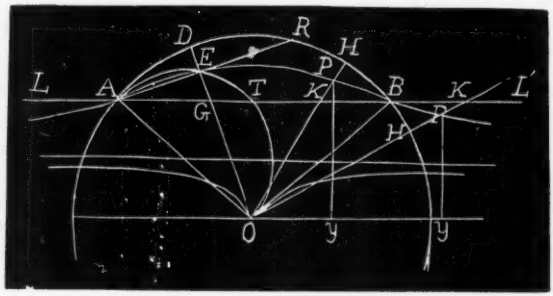
Also, $n=\sqrt{(x^2+y^2)}-r$, and substituting this value of n in (1) gives

$$\frac{y^2}{d^2}[2\sqrt{(x^2+y^2)}-r]^2=x^2+y^2, \tag{3}$$

which is the equation of the required locus.

For convenience of discussion, (3) may be put in form

$$y^2[2\sqrt{(x^2+y^2)}-r-d][2\sqrt{(x^2+y^2)}-r+d]=d^2x^2, \tag{4}$$



It is evident that the curve has two branches which meet at $\pm\infty$ on the asymptote whose equation is $y=\frac{1}{2}d$. n is regarded as negative when *in* the circle, and positive when *outside* of the circle.

The ceroid may be used to trisect any angle. For if AOB is a given angle, and ATO a semi-circle on AO as diameter, and E is the point common to the circumference ATO and the ceroid, the angle AOE is one third of the angle AOB.

Proof. $DE=GE$, for E is on the ceroid.

The angle AEO is a right angle. Hence angle DAE=angle GAE. Then arc AD=arc DR=arc RB. Therefore the angle AOD=one third of the angle AOB.

NOTE ON THE POSTULATE THAT A PART IS EQUIVALENT TO THE WHOLE.

By DR. G. A. MILLER.

The quadratic equations considered in elementary algebra are generally written in one of the following two forms:

$$ax^2+bx+c=0\ldots A,$$
$$ax^2+2bx+c=0\ldots B,$$

where a, b, c are real numbers and a is supposed to differ from zero. An equation of form A is completely determined by the two ratios b/a, c/a. If these two ratios are regarded as the co-ordinates of a point and plotted in the ordinary way with respect to a Cartesian system, every finite point in the plane corresponds to a quadratic equation and every quadratic equation corresponds to such a point. The necessary and sufficient condition that two quadratic equations have different roots is that they correspond to different points. According to this well known interpretation the points of the plane are regarded as representatives of quadratic equations.

The condition that the two roots of an equation of form A are equal to each other is $\frac{b^2}{a^2} = 4\frac{a}{c}$. Representing $\frac{b}{a}$ by x and $\frac{a}{c}$ by y this condition becomes $x^2=4y$. Hence the points on the parabola whose equation is $x^2=4y$ represent the quadratic equations of form A whose roots are equal. The points within this parabola, *i. e.*, on the concave side of this curve, represent the quadratic equations whose roots are imaginary, while those on the outside of the parabola represent the equations whose roots are real and distinct. The fact which we desire to emphasize in this connection is that the totality of quadratic equations with real coefficients and imaginary roots can be put into a (1, 1) correspondence with the points within the parabola $x^2=4y$. In other words, the number of the points within this parabola is equivalent to the number of quadratic equations having real coefficients and imaginary roots.

If we suppose that all the equations under consideration are represented by form B, those which have equal roots will be represented by the points of the parabola $x^2=y$. From this it follows that there is a (1, 1) correspondence between the totality of quadratic equations with real coefficients and imaginary roots, and the points within the parabola $x^2=y$. As the number of the points within each of the two parabolas $x^2=4y$ and $x^2=y$ is equivalent to the number of quadratic equations with real coefficients and imaginary roots, the two parabolas must contain the same number of points; *i. e.*, the number of points in one of these parabolas is equivalent to the number of points in the other. As the parabola $x^2=4y$ lies entirely within the one whose equation is $x^2=y$ except where they touch each other, a part of the number of points contained in the parabola $x^2=y$ is equivalent to the total number of these points.

While the above is only one of an indefinite number of illustrative examples of the reasonableness of the postulate that a part of an infinite number is equivalent to the whole, yet it seems to deserve special emphasis in view of its contact with very fundamental matters. Such illustrations seem desirable to prepare the way for the modern definitions of an infinite number; viz., An infinite number is equivalent to a part of itself, or an infinite number remains unchanged if unity is added to it.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Remarks on Problem 179 by TSURUICHI HAYASHI, Koto Shihan Gakko, Tokyo, Japan.

The following question proposed by Dr. L. E. Dickson, The University of Chicago, remains unsolved in Vol. XII, 1905, p. 238:

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0.$$

I think that the coefficients of x^2 and x must be interchanged and thus the equation must become

$$x^5+px^2+qx+\tfrac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0.$$

If so, the roots are represented by

$$-\omega_\lambda\sqrt[5]{\frac{q^2}{5p}}+\omega_\lambda^3\sqrt[5]{\frac{p^3}{25q}}$$

where $\lambda=1, 2, 3, 4, 5$, and $(\omega_\lambda)^5=1$.

Remark by the PROPOSER.

The problem was printed incorrectly; it should have read $x^5+px^2+qx+\ldots$, with the letters p, q in their natural order. Since p is of the third degree in the roots, and q of the fourth, the constant term is of the fifth degree, as should be the case.

Mr. T. Hayashi's solution of the corrected equation has, doubtless by an oversight in copying, the terms ω and ω^3 interchanged.

Solution. Since the terms x^4 and x^3 are lacking, the simplest expressions to assume for the roots are

$$r_\lambda=\omega_\lambda^3 A+\omega_\lambda B \quad (\lambda=1, \ldots, 5;\ \omega_\lambda^5=1).$$

Then by $\sum_{\lambda=1}^{5}\omega_\lambda^r=0$ for $r=1, \ldots, 4$, we have for $S_t=\Sigma r_\lambda^t$,

$$S_1=S_2=0, \quad S_3=15AB^2, \quad S_4=20A^3B, \quad S_5=5A^5+5B^5.$$

But for $x^5+px^2+qx+r=0$,

$$S_1=S_2=0, \quad S_3=-3p, \quad S_4=-4q, \quad S_5=-5r.$$

Hence, $A=-\sqrt[5]{\frac{q^2}{5p}}$, $B=\sqrt[5]{\frac{p^3}{25q}}$, $r=\frac{q^2}{5p}-\frac{p^3}{25q}$.

276. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $x_1, x_2, \ldots, x_n$ be unequal, and $f(x)$ be a rational integral function of degree $\geqslant n-2$, then shall

$$\sum_{r=n-1}^{r=1} \frac{f(x_r)}{(x_r-x_1)\ (x_r-x_2)\ldots(x_r-x_n)}=0.$$

Solution by the PROPOSER.

The left hand side written at length is

$$\frac{f(x_1)}{(x_1-x_2)\ (x_1-x_3)\ldots(x_1-x_n)}+\frac{f(x_2)}{(x_2-x_1)\ (x_2-x_3)\ldots(x_2-x_n)}+\ldots$$

$$+\frac{f(x_n)}{(x_n-x_1)\ (x_n-x_2)\ldots(x_n-x_{n-1})}.$$

Let $\frac{f(x)}{(x-x_1)\ (x-x_2)\ldots(x-x_n)} \equiv \frac{A_1}{x-x_1}+\frac{A_2}{x-x_2}+\ldots+\frac{A_n}{x-x_n}$.

Then $A_1, A_2, A_3, \ldots, A_n$

$$=\frac{f(x_1)}{(x_1-x_2)\ (x_1-x_3)\ldots(x_1-x_n)}, \quad \frac{f(x_2)}{(x_2-x_1)\ (x_2-x_3)\ldots(x_2-x_n)}, \ldots,$$

$$\frac{f(x_n)}{(x_n-x_1)\,(x_n-x_2)\ \ldots\ (x_n-x_{n-1})}.$$

Hence, $f(x) \equiv \Sigma A_1 (x-x_2)(x-x_3)\ldots(x-x_n)$ = polynomial of degree $\geqslant n-2$.
Hence, $\Sigma A_1 = 0$.

This problem, as we thought, proves to be similar to Ex. 4, p. 319, 3rd Edition of Burnside and Panton's *Theory of Equations*. ED. F.

GEOMETRY.

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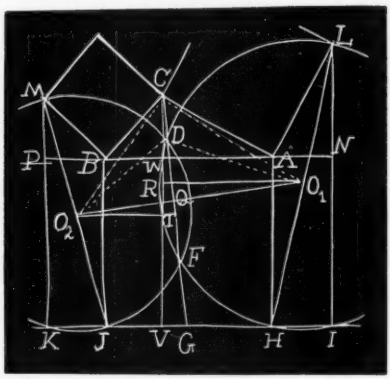
311. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Dallas High School, Dallas, Texas.

Triangle ABC is obtuse-angled at C; x, y, z are squares on the sides AC, CB, BA; LH and MJ are lines joining adjacent sides of x, z and y, z. The common chord of the circles on LH and MJ as diameters passes through C and the mid-point of HJ.

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Let ABC be the given triangle with the squares $ABJH$, $BM-C$, $AL-C$ constructed on the sides AB, BC, and AC, respectively. Join M and J and L and H. On LH and MJ as diameters describe the circles whose centers are O_1 and O_2, respectively, the circles intersecting in the points D and F. Draw LI parallel to AH, and meeting JH produced in I. Similarly, draw MK. I and K are on the circumferences of the circles, since angles MKJ and LIH are right angles.



Produce AB to meet MK in P and LI in N. Draw CV perpendicular to JH, and O_1R and O_2T each perpendicular to CV. Draw the common chord DFG and join O_1 and C, O_2 and C, O_1 and D, O_2 and D, D and C, and O_1 and O_2, the line O_1O_2 intersecting DF in Q. Then, from the right triangles O_1QD and O_2QD, we have

$$O_1D^2 - O_1Q^2 = O_2D^2 - O_2Q^2, \text{ or } O_1D^2 - O_2D^2 = O_1Q^2 - O_2Q^2 \ldots (1).$$

Now, if FDC is a straight line, we must have

$$O_1C^2 - O_1Q^2 = O_2C^2 - O_2Q^2, \text{ or } O_1C^2 - O_2C^2 = O_1Q^2 - O_2Q^2 \ldots (2).$$

Hence, we must have

$$O_1C^2 - O_2C^2 = O_1D^2 - O_2D^2 \ldots (3)$$

Now, $O_1C^2 - O_2C^2 = (O_1R^2 + CR^2) - (O_2T^2 + CT^2)$

$= (O_1R + O_2T)(O_1R - O_2T) - (CT + CR)(CT - CR)$

$= (\frac{1}{2}AN + AW + WB + \frac{1}{2}BP)(\frac{1}{2}AN + AW - WB - \frac{1}{2}BP)$

$- (CV - RV + CV - VT)(CV - VT - CV + RV)$

$= \frac{3}{2}(AW + WA + CW)(AW - WB) - [2CW - (RV + TV)](RV - TV),$

since $AW = CW = BP$, $= \frac{3}{2}(AW + WB + CW)(AW - WB) - [2CW + 2AB)$

$- \frac{1}{2}(LN + NI + MP + PK)][\frac{1}{2}(LN + NI) - \frac{1}{2}(MP + PK)]$

$= \frac{3}{2}(AW + WB + CW)(AW - WB) - \frac{1}{2}(2CW + \frac{3}{2}AB)(AW - WB)$

$= \frac{3}{4}(AW^2 - WB^2) = \frac{3}{4}(AC^2 - BC^2).$

Also, $O_1D^2 - O_2D^2 = (O_1D + O_2D)(O_1D - O_2D) = \frac{1}{2}(LN + NI + MP + PK)$

$\times \frac{1}{2}(LN + NI - MP - PK) = \frac{1}{2}(AW + WB)(AW - WB),$

since $LN = AW$ and $MP = WB$, $= \frac{3}{4}(AW^2 - WB^2) = \frac{3}{4}(AC^2 - BC^2).$

Hence, (3) is true and, therefore, GDC is a straight line.

To prove that G is the mid-point of HJ, we have, from the secants GD, GK, and GI, $GK.GJ=GD.GF=GI.GH$, or $GK.GH=GI.GJ$.

Hence, $GK+GH : GH=GI+GJ : GJ$, or $KH : GH=IJ : GJ$.

$\therefore GH=GJ$, since $HI=JK$ and, therefore, $KH=IJ$.

Also solved analytically by G. B. M. Zerr and J. Scheffer. Professors Zerr and Scheffer obtain the equation of the common chord of the two circles and found that its equation is satisfied by the co-ordinates of C and G.

CALCULUS.

236. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve the partial differential equation, $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial u}{\partial x}$.

I. Solution by C. EBEN STROMQUIST, Princeton, N. J.

A more general problem, of which the above is a particular case, has been discussed by Darboux (*Leçons sur la Theorie Generale Surfaces*, Vol. 3, p. 53). The solution in this case follows.

Differentiating the given equation with respect to x and setting

$$M=\frac{\partial^2 u}{\partial x^2},\ \text{yields}\ x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}=0\ldots(2).$$

This is a linear differential equation of the first order in M of which a most general solution is* $M=W(x/y)$, where W is an arbitrary function in the argument x/y. Hence

$$w=\int_{x_0}^{x}\int_{x_0}^{x} W\,dx\,dx+x\,w_1(y)+w_2(y)\ldots(3),$$

where w_1 and w_2 are functions of y alone which must be so restricted that w satisfies the given equation.

Since the differential equation from which u is obtained is equation (1) differentiated with respect to x, the result of substituting u in (1) must yield at most a function of y alone. If Y be the function resulting from substituting $u_0=\int_{x_0}^{x}\int_{x_0}^{x} W\,dx\,dx$ in (1), it is readily seen that $Y=0$, and hence that the result of substituting u in (1) gives

$$y\frac{\partial w_1}{\partial y}-w_1=0\ldots(4).$$

Integrating gives $w_1=cy$, whence c is an arbitrary constant.

Substituting in (3), a most general solution of (1) is given by

*See, for instance, Jordan, *Cours d'Analyse*, Vol. 3, p. 314.

$$u=\int_{x_1}^{x}\int_{x_0}^{x} W(\frac{x}{y})dx\,dx+cxy+w_2(y),$$

where W and w_2 are arbitrary functions in x/y and y, respectively, and c is an arbitrary constant.

II. Solution by GEORGE W. HARTWELL, Columbia University, New York City.

Let $\frac{\partial u}{\partial x}=p$. Then the equation will become

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}=p.$$

Solving this by the method of Lagrange we have

$$p=y\,\phi(\frac{x}{y})=\frac{y^2}{y}\,\phi\,(\frac{x}{y}).$$

Let $\phi(\frac{x}{y})$ be the derivative of $\psi(\frac{x}{y})$ with respect to $\frac{x}{y}$. Then

$$\frac{1}{y}\,\phi\,(\frac{x}{y})=\frac{\partial}{\partial x}\,\psi\,(\frac{x}{y}).\quad\therefore\frac{\partial u}{\partial x}=y^2\frac{\partial}{\partial x}\,\psi\,(\frac{x}{y}).$$

Integrating, $u=y^2\,\psi\,(\frac{x}{y})+\chi(y)$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Multiply the equation by x and let $x=e^v$, $y=e^w$.

Then $\frac{d^2u}{dv^2}+\frac{d^2u}{dvdw}=2\frac{du}{dw}$. Let $\frac{d}{dv}=D$, $\frac{d}{dw}=D'$. Then $D(D+D'-2)u=0$.

Let $u=e^{av+bw}$. $\therefore\frac{du}{dv}=au$, $\frac{du}{dw}=bu$.

$\therefore a(a+b-2)=0$, therefore, $a=0$, and $a=2-b$.

$\therefore u=\Sigma Ae^{bw}+e^{2v}\Sigma Be^{b(w-v)}=F(w)+e^{2v}f(w-v)=F(y)+x^2f(y/x)$, where F and f are arbitrary.

Also solved by the Proposer.

MECHANICS.

197. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

Suppose that a primary planet and its satellite revolve with uniform angular velocity in circular orbits in the same plane. What relation must hold between the radii of their orbits and their angular velocities in order

that the curve traced by the satellite shall be everywhere concave to the sun? Apply to the earth-moon system to prove that the moon's path is always concave to the sun.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let m=sun's mass, m_1=primary's mass, R=distance of primary from sun, r=distance of satellite from primary, v=velocity of primary around the sun at distance R, v_1=velocity of satellite around the sun at distance R, v_2=velocity of satellite around the primary at distance r.

Then $v^2/R : v_1^2/r = m : m_1$; $v : v_2 = \sqrt{m} : \sqrt{m_1}$; $v_2 : v_1 = \sqrt{r} : \sqrt{R}$.

$\therefore v : v_1 = \sqrt{(mr)} : \sqrt{(m_1 R)}$, or $v/R : v_1/r = \sqrt{(mr^3)} : \sqrt{(m_1 R^3)}$, the ratio of the angular velocities of primary and satellite in their respective orbits.

Hence, the path of the satellite will be looped, cusped, or direct throughout if

$$\sqrt{(m_1 R^3/mr^3)} >= < R/r; \text{ or } m_1 R >= < mr, \text{ or } m_1/m >= < r/R.$$

From these, we learn that the path of the satellite will be partly convex, just fail of being convex at perihelion, or be concave, if

$$m_1 R^3/mr^3 >= < R/r; \text{ or } m_2 R^2 >= < mr^2; \text{ or } m_1/m >= < r^2/R^2;$$
$$\text{or } \sqrt{(m_1/m)} >= < r/R.$$

For the earth and moon, $m/m_1 = 322{,}700 = (568)^2$, $R = 92{,}000{,}000$. Hence, if the moon were $92{,}000{,}000/322{,}700 = 285$ miles from the earth, it would travel in a cusped epicycle. If $92{,}000{,}000/568 = 162{,}000$ miles from the earth, the epicycle would be convex. As the actual distance is 238,828 miles, it is always concave.

Also solved by A. H. Holmes.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

141. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given that the highest factor of a prime p contained in $m!$ is p^{m-s}; find general expressions involving p and m and s, from which, when a solution is possible, m can be determined when s is a given integer and p is a given prime. Is it then possible in any case to have more solutions than one?

No solution has been received.

142. Proposed by DR. L. E. DICKSON, The University of Chicago.

Let n be an integer >1 and set $p = n(n-1)+1$. Required n integers whose $n(n-1)$ differences are congruent (modulo p) to the numbers $1, 2, \ldots, p-1$. Exhibit at least for $n = 3, 4, 5$, all inequivalent sets of solutions where a set $a_1, a_2, \ldots, a_n$ is called equivalent to the set $m(a_1 - d), m(a_2 - d), \ldots, m(a_n - d)$, for any integers m and d (m not divisible by p).

Solution by DR. C. R. MacINNES, Princeton, N. J.

As Dr. Safford pointed out in discussing No. 132, the problem is equivalent to the following: Find n numbers such that when we add consecutive ones in every possible way, we get all the numbers from 0 to $n(n-1)$. Having any solution of the original problem, without loss of generality we may arrange the numbers in order of magnitude and so get n intervals which add up to $n(n-1)+1$, two of which will be 1 and 2. Such sets can be searched for systematically.

For $n=3$, there is only one set of intervals, 1, 2, 4.

For $n=4$, there are two sets, 1, 2, 6, 4, and 1, 3, 2, 7. But these are equivalent, since multiplying the second set by 2 reproduces the first.

For $n=5$, there is only one set, 1, 5, 2, 10, 3.

For $n=6$, there are five sets, 1, 3, 2, 7, 8, 10; 1, 3, 6, 2, 5, 14; 1, 2, 5, 4, 6, 13; 1, 7, 3, 2, 4, 14; and 1, 2, 7, 4, 12, 5. But these are equivalent, since, if we multiply the solutions corresponding to the first four by 4, 12, 23, and 28, respectively, and add 10, 14, 3, and 3, respectively, we get the last one.

For $n=7$, there is no solution.

For $n=8$, there are six sets of intervals, all of them equivalent to 1, 2, 10, 19, 4, 7, 9, 5.

The original problem, then, has no solution for $n=7$, a unique solution for other values of n up to 8.

n	*Solution.*
3	0, 1, 3.
4	0, 1, 3, 9.
5	0, 1, 6, 8, 18.
6	0, 1, 3, 10, 14, 26.
8	0, 1, 3, 13, 32, 36, 43, 52.

AVERAGE AND PROBABILITY.

179. Proposed by HENRY HEATON, Bellfield, N. D.

Through every point of the circumference of a given circle, chords are drawn in every possible direction. What is their average length?

No solution has been received.

180. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

There are n numbers in a box numbered from 1 to n. A number is drawn and replaced n times. Show that on the average the number of repeats is $\left(\frac{n-1}{n}\right)^n n$.

No solution has been received.

181. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

At a sea-side excursion for x men there are boats enough for q men, and carriages enough for z. But p do not care for driving, and q would feel indifferently comfortable on the water, while the rest do not care either way. Each man has what he prefers as long as a seat is left for him in carriages or boats, and those who do not care either way choose at random. Find the chance that all will be satisfied.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$x<z+q$, $p<q$. Let c=required chance. $x-p-q$ choose at random.

After the $p+q$ persons are satisfied there are still left $q-p$ boats and $z-q$ carriages or $z-p$ conveyances left for the random choosers to select from. $x-p-q$ things can be selected from x things in

$$N=\frac{x!}{(x-p-q)!\,(p+q)!} \text{ ways.}$$

$x-p-q$ things can be selected from $z-p$ things in

$$n=\frac{(z-p)!}{(x-p-q)!\;(z+q-x)!} \text{ ways.}$$

Then $c=\dfrac{n}{N}=\dfrac{(z-p)!\;(p+q)!}{x!\;(z+q-x)!}$.

182. Proposed by L. MORDELL, Philadelphia, Pa.

Out of n straight lines whose lengths are 1, 2, 3, 4, ..., n inches, respectively, the number of ways in which 4 may be chosen which will form a quadrilateral in which a circle may be inscribed is $\frac{1}{48}[2n(n-2)(2n-5)-3+3(-1)^n]$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Taking the first four numbers we get one possible case; taking five, 3; taking six, 7 cases; etc. Thus we have the series 1, 3, 7, 13, 22, 34, 50, 70, 95, 125, ..., of which we have to find the general term. If n is an *even* number, we have the series 1, 7, 22, 50, 95, ..., of which we find $a_0=1$, $\triangle a_0=6$, $\triangle^2 a_0=9$, $\triangle^3 a_0=4$, $\triangle^4 a_0=\triangle^5 a_0\ldots=0$. The number of terms is $\frac{1}{2}n-1$.

$\therefore y_n=\frac{1}{24}n(2n^2-9n+10)=\frac{1}{24}n(n-2)(2n-5)$.

If n is an *odd* number, we have the series 3, 13, 34, 70, 125, ..., of which $a_0=3$, $\triangle a_0=10$, $\triangle^2 a_0=11$, $\triangle^3 a_0=4$. Thus, the number of terms being $\dfrac{n-3}{2}$, we find $\frac{1}{24}n(2n^2-9n+10)-\frac{3}{24}=\frac{1}{24}n(n-2)(2n-5)-\frac{3}{24}$.

Both formulae may be condensed into $y_n=\frac{1}{48}[2n(n-2)(2n-5)-3+3(-1)^n]$.

Also solved by G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

284. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of equations:

$$\begin{cases} x^k+y^k=a_k \\ x^l+y^l=a_l \end{cases} \quad (k,\ l \text{ distinct positive integers})$$

in general and for particular values of $(k,\ l;\ a_k,\ a_l)$.

285. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of equations:

$$\begin{cases} x^k+y^k+z^k=a_k \\ x^l+y^l+z^l=a_l \\ x^m+y^m+z^m=a_m \end{cases} \quad (k,\ l,\ m \text{ distinct positive integers})$$

in general and for particular values of $(k,\ l,\ m;\ a_k,\ a_l,\ a_m)$.

286. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of n equations in $x_1,\ x_2,\ \ldots,\ x_n$:

$$\begin{aligned} &{x_1}^{k_1}+{x_2}^{k_1}+\ldots+{x_n}^{k_1}=a_1 \\ &{x_1}^{k_2}+{x_2}^{k_2}+\ldots+{x_n}^{k_2}=a_2 \\ &\quad\vdots\qquad\quad\vdots\qquad\quad\vdots\qquad\quad\vdots \\ &{x_1}^{k_n}+{x_2}^{k_n}+\ldots+{x_n}^{k_n}=a_n \end{aligned}$$

where the $k_1,\ \ldots,\ k_n$ are n distinct positive integers, and the $a_1,\ \ldots,\ a_n$ are n given numbers.

GEOMETRY.

317. Proposed by J. STEWART GIBSON, Department of Physics, Wadleigh High School, New York City.

Find the locus of the vertices of the parabolas described by particles thrown off a uniformly revolving circumference.

318. Proposed by G. W. GREENWOOD, M. A., Roanoke College, Salem, Va.

Is it possible by a straight edge and sect carrier, *i. e.*, without the use of a circle, to construct a mean proportional to two given sects?

CALCULUS.

240. Proposed by L. MORDELL, Philadelphia, Pa.

Show that the osculating conic of the catenary $y=c\cosh\frac{x}{c}$ at the point for which $y=\frac{c\sqrt{10}}{2}$ is a parabola.

241. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

Differentiate

$$y=1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\text{etc.}}}}}$$

MECHANICS.

203. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A train weighing $T(=80)$ tons runs first eastward and then westward in latitude $\lambda(=40°)$ at a velocity $v(=45)$ miles an hour. Find the difference between the pressures on the ground in the two cases.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

146. Proposed by PROFESSOR JOSE DE J. CORONADO, Halapa, Vericruz, Mexico.

Find two numbers whose difference is equal to the difference of their cubes.

AVERAGE AND PROBABILITY.

189. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

(a) Lines are drawn from the vertices of a triangle through a random point within it. Find the average area of the triangle formed by joining the points of intersection of these lines with the opposite sides. (b) Lines are drawn from the vertices to points taken at random in the opposite sides of a triangle. Find the average area of the triangle formed by the intersections of these lines.

NOTES AND NEWS.

Mr. S. A. Corey was elected a member of the American Mathematical Society April 27, and was also recently elected a member of the Circolo Matematico di Palermo.

Cardinal Maffi, raised to the purple lately, is a mathematician of considerable note; he started the "Rivista di scienze fisiche e matematiche" and took an active part in the founding of the Italian Catholic Society for Scientific Studies, and in 1903 was made president. J. H. M.

The University of Pennsylvania offers the following advanced courses in mathematics at the summer session, July 8 to August 17, 1907, each course consisting of thirty lectures: Higher Analytic Geometry, by Prof. E. S. Crawley; Definite Integrals, by Prof. I. J. Schwatt; Theory of Functions of a Complex Variable, by Prof. G. H. Hallett; Differential Equations, by Dr. F. H. Safford.

The following courses are to be offered in Collegiate Mathematics at the Summer Session of the University of Illinois: By Dr. Sisam: Modern Geometry, Plane Analytical Geometry. By Dr. Dodd: Functions of Real Variables, Plane Trigonometry. By Mr. Ponzer: Integral Calculus, Teachers' Course in Pedagogy of Secondary Mathematics. By Mr. Emmons: Theory of Equations, College Algebra. All courses are given daily and are equivalent to corresponding courses offered during the regular college year.

The following advanced courses in Mathematics will be offered at the University of Pennsylvania during the year 1907-08: By Professor E. S. Crawley: Solid Analytic Geometry, two hours; Higher Plane Curves, three hours. By Professor G. E. Fisher: Theory of Functions of a Complex Variable, first half year, three hours; Elliptic Functions, second half year, three hours. By Professor I. J. Schwatt: Definite Integrals, three hours. By Professor G. H. Hallett: Lie's Theory of Continuous Groups, first half year, three hours; Galois' Theory of Algebraic Equations, second half year, three hours. By Dr. F. H. Safford: Curvilinear Co-ordinates, three hours. By Dr. O. E. Glenn, Higher Algebraic Equations, two hours.

The following courses in mathematics are to be offered at the University of Wisconsin during the summer session, 1907: By Professor Van Vleck: Solid Geometry, admission credit; Geometry, a general survey of the progress of geometry, two hours' credit; Theory of Point Sets, two hours' credit. By Professor Dowling: Introduction to Higher Plane Curves, one hour's credit; Differential and Integral Calculus, two hours' credit; Plane Trigonometry, two hours' credit. By Professor Skinner: Elementary Algebra, admission credit or two hours' credit. By Assistant Professor Mason: Analytic Geometry, two hours' credit; Application to Mechanics, one hour's credit; Number Concept and Geometric Construction, one hour's credit.

The followlng courses in mathematics are to be offered at the University of Michigan during the summer session, 1907: By Professor Beman: Differential Equations, two hours' credit; Geometry and Algebra, a course for teachers, two hours' credit. By Professor Markley: Theory of Functions of a Complex Variable, two hours' credit; Projective Geometry, two hours' credit; College Algebra, two hours' credit. By Assistant Professor Glover: Elementary Algebra, admission credit; Theory of Annuities and Insurance, two hours' credit. By Dr. Running: Analytic Mechanics, two hours' credit; Trigonometry, two hours' credit. By Dr. Field: Differential Calculus, two hours' credit; Integral Calculus, two hours credit. By Mr. Escott: Plane Geometry, admission credit; Analytic Geometry, four hours' credit.

BOOKS AND PERIODICALS.

Elements of the Infinitesimal Calculus. By G. H. Chandler, M. A., Professor of Applied Mathematics, McGill University, Montreal. Third Edition, rewritten. 12mo Cloth, vi+319 pages, 146 figures. Price, $2.00. New York: John Wiley & Sons.

This work is intended as an introductory manual of the Calculus for beginners generally and students of engineering particularly. The author recognizes the Theory of Limits as the logical foundation of the Calculus, but in his book he aims to accustom the reader to the principles and processes which are used in practical operations. The arrangement of topics is somewhat out of the usual order. Thus the differentials of Hyperbolic Functions are discussed immediately after the differentials of the ordinary functions. Also integration is taken up early, page 93. Some subjects not usually treated in elementary texts find a brief treatment in this work. Thus the Intrinsic Equations of Curves, Fourier Series, and Differential Equations are treated, Fourier Series occupying pages 228-238 and Differential Equations 259-286. Elliptic Integrals are also briefly discussed. Many of the problems are related to practical application and there are many illustrating the various subjects under discussion. At the end of the book there are the following tables: (1) Powers, Neaperian Logarithms, etc. (2) Circular Functions; Hyperbolic Functions; Lambda Functions: Gamma Functions; First Elliptic Integrals; and Second Elliptic Integrals. B. F. F.

Higher Mathematics for Students of Chemistry and Physics, with Special Reference to Practical Work. By J. W. Mellor, D. Sc. Second Edition, enlarged. 8vo Cloth, xxi+631 pages. Price, $4.50 net. New York: Longmans, Green, & Co.

The author has rendered a good service to the students of chemistry and physics by preparing for their use this excellent work. In the establishment of principles and rules, the author has kept constantly in mind the class of students for which he was preparing his book, and thereby he has avoided the extreme logical rigor and and subtle formalities on the one hand, while at the same time no loose and illogicel reasoning is permitted, on the other.

The student will in the study of this work get a good grounding in the Differential and Integral Calculus, Infinite Series, Differential Equations, Theory of Error, Calculus of Variations, etc. While some of the problems belong to the domain of pure mathematics and are to be found in many text-books, yet the greater number are based upon measurements, etc, recorded in current scientific journals.

The style of type, printing, and binding are excellent, and the work is one that will commend itself favorably to all classes of mathematical students. B. F. F.

The Electron Theory. A Popular Introduction to the New Theory of Electricity and Magnetism. By E. E. Fournier D'Albe, B. Sc. (London), A. R. C. Sc., Compiler of "Contemporary Electrical Science" with a Preface, by G. Johnstone Stoney, M. A., Sc. D., F. R. S. 8vo Cloth. xxiii+311 pages. Price $1.50 net. New York: Longmans, Green, & Co.

In this work is set forth in popular language the latest theory regarding the nature of matter. All students of science unacquainted with the most recent advances in Electricity and Magnetism, will want this book. B. F. F.

Wellcome's Photographic Exposure Record and Diary. Price, 50 cents. London and New York: Burroughs, Wellcome, & Co.

This little book gives a clear explanation of the underlying principles of exposure and puts them into practice by means of a very simple mechanical calculator which indicates the correct exposure. The book is a compact compendium of photographic information. In addition it provides a pocket note-book, a diary and ruled pages for recording exposures. This little book will be of great service to the amateur photographer. B. F. F.

New Elementary Arithmetic. By George Wentworth, Author of Text-Books in Mathematics. 8vo Cloth, 232 pages. Boston and Chicago: Ginn & Co.

This book is designed to be taken up not later than the last half of the second school year. The work, in addition to well selected material for the pupils, contains also some good suggestions for the teacher. B. F. F.

Theoretical Mechanics. By J. H. Jeans, Fellow of Trinity College, Cambridge (England), and Professor of Applied Mathematics in Princeton University. 8vo Cloth, 364 pages. Illustrated. List price, $2.50; mailing price, $2.65. Boston and Chicago: Ginn & Co.

This book is intended to supply a one year's course for students beginning the study of mechanics. It treats of the general principles of dynamics, the laws of motion, statics, and dynamics of a particle of a rigid body. The treatment of the subject is lucid and each principle is illustrated by a series of practical examples. Numerous well selected exercises and problems are inserted for solution by the student. B. F. F.

Plane Geometry. By Edward Rutledge Robbins, A. B., Senior Mathematical Master, The William Penn Charter School. 8vo, Cloth sides, Leather back, 254 pages. New York, Cincinnati, and Chicago: American Book Co.

Among the reasons given by the author in preparing this work are: To present a book that has been written for the pupil; to stimulate his mental activity; to present a text that will be clear, consistent, teachable, and sound; and to explain rather than formally demonstrate the simple fundamental truths. The diagrams are good, the typographical and mechanical execution of the book is splendid, and the book is well written. A large number of original exercises is inserted. B. F. F.

Physics. By Charles Riborg Mann, the University of Chicago, and George Ransom Twiss, the Central High School, Cleveland. 8vo Cloth, 458 pages. Chicago: Scott, Forsman, & Co.

This book is so written and the subject matter presented in such a way that the attention of the pupil is held to the subject under discussion. The illustrations are new, being, for the most part, pictures of real objects. The most recent discoveries in Physics are here clearly described and the underlying principles fully explained. This work will go far towards stimulating a greater interest in the study and teaching of Physics. B. F. F.

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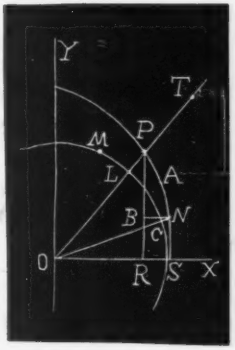
ON DIVIDING AN ANGLE INTO PARTS HAVING THE RATIOS OF ANY GIVEN STRAIGHT LINES.*

By REV. R. D. CARMICHAEL, Presbyterian College, Anniston, Alabama.

The solution of this problem will be here effected by aid of the locus of the polar equation $\rho \sin \theta = m\theta$. In a similar manner it may be carried out by means of the locus of $\rho \cos \theta = m\theta$. We shall take the special case $m=1$ of the first equation, thus giving $\rho \sin \theta = \theta$.

It is desirable to have a method of constructing the curve by continuous motion. We proceed in the following manner: Construct a material circle with center O and radius unity, as in the figure, and let it be fixed to the plane of the paper. Let O be the origin and OX the polar axis. Let P be any point on the curve, and let OPT be a straight bar pivoted to the paper at O and free to move without carrying the circle with it. Draw PR perpendicular to OX; now we have $OP=\rho$, $\angle POR=\theta$, and $PR=\rho \sin \theta$. Then from the equation of the curve, $PR=\theta$. That is, PR is numerically equal in length to the arc LCS. Now fasten one end of a cord at some point Q in the circumference of the circle and pass it any convenient number of times around the circle in the direction QLS. On the last round let the free end of the cord pass through a small ring fastened to OT at L. Then let it pass around a roller at T and back through another small ring at P, with its free end taking the direction PR perpendicular to OX at R, the extremity of the cord being at R. Take now another chord of length $LT+PT$, with its ends attached to the rings at L and P, the cord also passing around the roller at T. As the line OT moves, let the roller at T so slide as to keep the last chord tightly stretched. (This may be accomplished by having T attached to a spring.) The purpose of this chord is to make the distance $LT+PT$ remain canstant as OT moves. If a pencil is also fastened



*Presented to the American Mathematical Society, February 23, 1907.